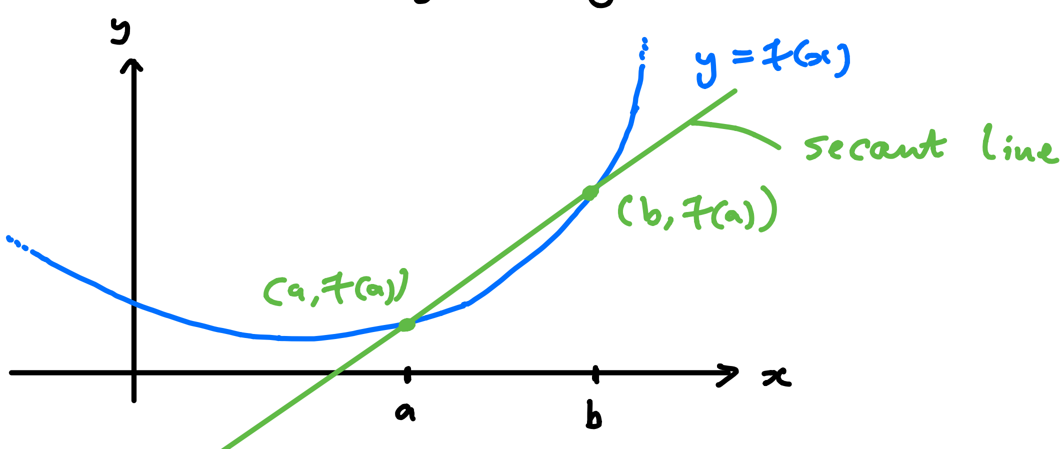
The Derivative

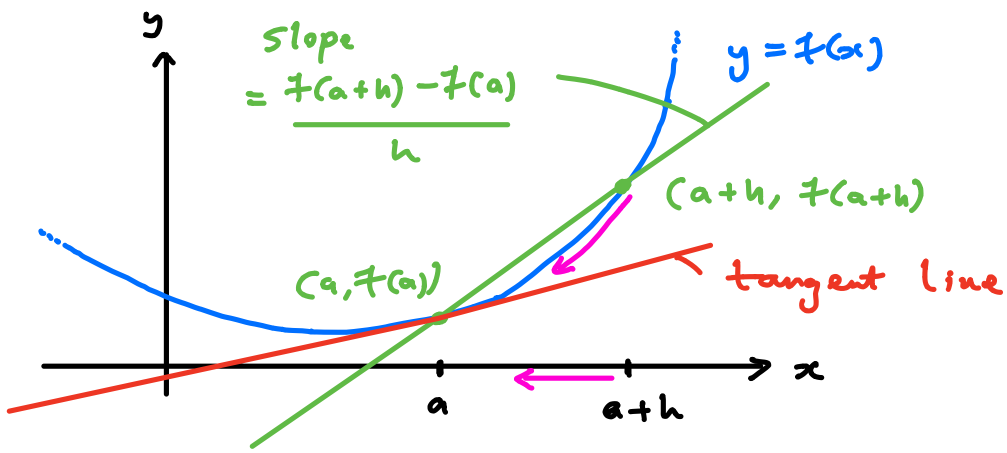
Average rate of change of f between $t=a$ and $t=b$ $= \dfrac{f(b)-f(a)}{b-a}$

Instantaneous rate of change of f at $x=a$ $= \lim\limits_{h\to 0}$ Average rate of change of f between $t=a$ and $t=a+h$ $= \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$

Q/: Can we interpret these quantities in terms of the graph $y=f(x)$?



Slope of Secant $= \dfrac{f(b)-f(a)}{b-a} =$ Average rate of change of f between $t=a$ and $t=b$



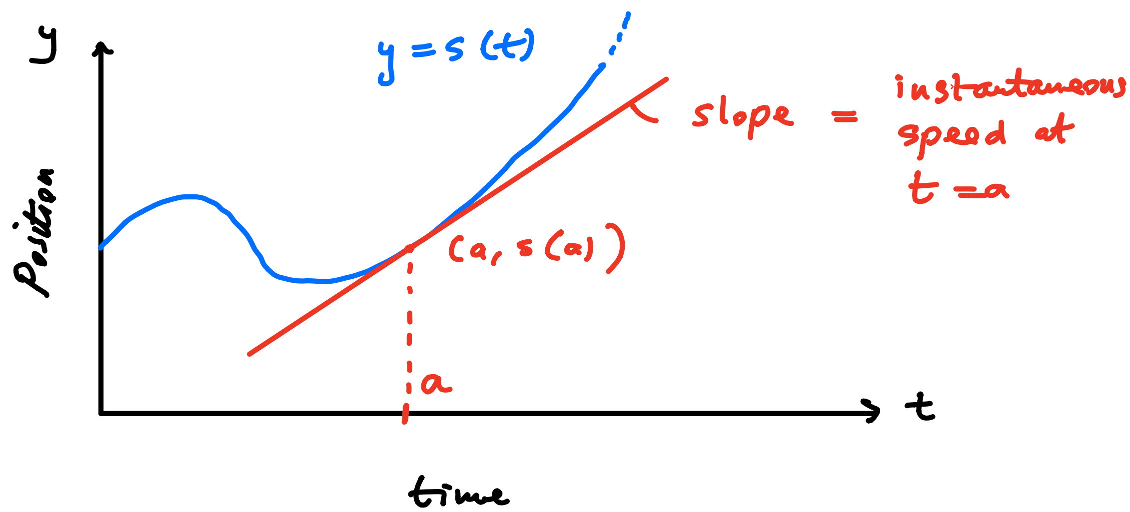
As h approaches 0, the secant line approaches the tangent line.

$\Rightarrow$ Slope of secant $\left(\frac{f(a+h)-f(a)}{h}\right)$ approaches slope of tangent line as h approaches 0.

Conclusion :

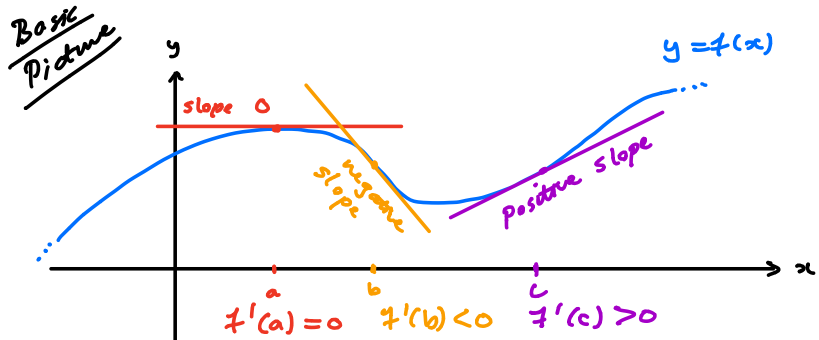
Instantaneous rate of change of f at $x=a$	$=$	Slope of tangent line at $(a, f(a))$

Example (Motion in straight line)



Notation : $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ (Assuming limit exists)

"f prime a"



Change of perspective : view a as a variable

Definition (the derivative)

The derivative of the function f, is the function f' (f prime) defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

called the difference quotient.

provided the limit exists.

$$\boxed{f'(x) = \text{slope of tangent line of graph at } (x, f(x)) = \text{Instantaneous rate of change of } f \text{ at } x}$$

Informally : the derivative measures the steepness of the graph at x.

Economics Examples $C(x)$, $R(x)$, $P(x)$ = cost, revenue, profit functions

$\Rightarrow$ $C'(x)$, $R'(x)$, $P'(x)$ = marginal cost, revenue, profit functions

Examples

1) $f(x) = x^2 \Rightarrow f'(x) = ?$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} 2x + h = 2x$$

Treat x like a constant here.

2/ $f(x) = \frac{1}{x} \Rightarrow f'(x) = ?$
$(x \neq 0)$

What is the equation of tangent line to $y = \frac{1}{x}$ at $x = 1$?

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h\to 0} \frac{x - (x+h)}{h\,x\,(x+h)} = \lim_{h\to 0} \frac{-1}{x\,(x+h)}$$

$$= \frac{-1}{x^2} \quad (x \neq 0)$$

$\Rightarrow f'(1) = \frac{-1}{1^2} = -1$ = slope of tangent line at $x = 1$

$f(1) = \frac{1}{1} = 1$

$\Rightarrow$ Tangent line has slope -1 and contains $(1, 1)$

$\Rightarrow y - 1 = -(x - 1)$

Conclusions : To calculate $f'(x)$

1/ Determine $f(x+h)$

2/ Take the difference $f(x+h) - f(x)$

3/ Write the difference quotient $\frac{f(x+h)-f(x)}{h}$ and simplify

4/ Take limit as h approaches 0.

Examples 1/ $f(x) = \sqrt{x} \Rightarrow f'(x) = ?$
$(x > 0)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} \qquad \left(\underbrace{(a-b)(a+b)}_{a^2 - b^2} \right)$$

$$= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}} \qquad (x > 0)$$

2/ $f(x) = x^3 + 1$. What is the equation of the tangent line to $y = f(x)$ at $x = 1$?

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^3 \cancel{+1} - x^3 \cancel{-1}}{h} = \lim_{h\to 0} \frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x^3}}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2$$

$f'(1) = 3 \cdot 1^2 = 3$

$f(1) = 1^3 + 1 = 2$ $\Rightarrow$ $y - 2 = 3(x-1)$ is equation of tangent.

<u>Definition</u> We say f <u>differentiable</u> at $x = a$ if $f'(a)$ exists, ie if both

1/ $f(a)$ exists

2/ $\displaystyle\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists

From both sides.

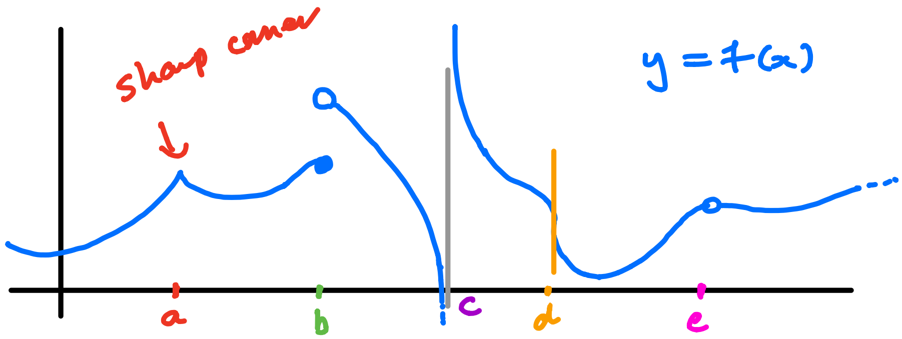
If either 1/ or 2/ fail we say f is non-differentiable at $x = a$.

f non-differentiable at $x = a$ $\Rightarrow$ There is no (non-vertical) tangent line at $(a, f(a))$

<u>Fact</u>:

f <u>differentiable at $x = a$</u> $\Rightarrow$ f <u>continuous</u> at <u>$x = a$</u>

Basic Picture



f is non-differentiable at a as there is a corner

f is non-differentiable at b as it is discontinuous

f is non-differentiable at c as it is discontinuous

f is non-differentiable at d as the tangent line is vertical.

f is non-differentiable at e as it is discontinuous

Examples 1) $f(x) = |x|$, $a = 0$

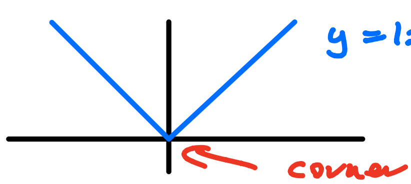
$$\frac{f(0+h) - f(0)}{h} = \frac{|h|}{h} = \begin{cases} 1 & \text{if } h > 0 \\ -1 & \text{if } h < 0 \end{cases}$$

$$\Rightarrow \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} 1 = 1$$

$$\neq \quad \Rightarrow \lim_{h \to 0} \frac{|h|}{h} \text{ DNE}$$

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} -1 = -1$$

$\Rightarrow$ $|x|$ is non-differentiable at 0.



2/ $f(x) = \sqrt[3]{x}$, $a = 0$

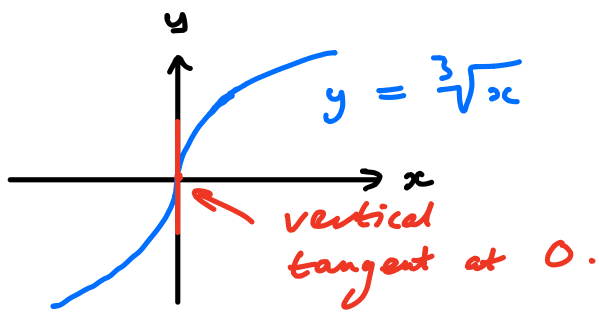
$$\frac{f(0+h) - f(0)}{h} = \frac{\sqrt[3]{h}}{h} = \frac{h^{1/3}}{h^1} = \frac{1}{h^{2/3}}$$

$$\lim_{h \to 0} h^{2/3} = \lim_{h \to 0} (h^{1/3})^2 = 0^+$$

+ because of the square

$$\Rightarrow \lim_{h \to 0} \frac{1}{h^{2/3}} = \infty \quad (DNE)$$

$\Rightarrow f(x) = \sqrt[3]{x}$ is non-differentiable at 0.



Conclusions

$$\boxed{\begin{array}{l} \displaystyle \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \neq \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \Rightarrow \text{Corner at } x \\[2ex] \displaystyle \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \underset{(DNE)}{\pm \infty} \Rightarrow \text{Vertical Tangent} \end{array}}$$